H28 問 4.

1)
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx.$$

a)
$$\begin{cases} f_1(x) = x, & -\pi \le x < \pi \\ f_1(x+2\pi) = f_1(x). \end{cases}$$

この $f_1(x)$ に対して，

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{1}{2}x^2\right]_{-\pi}^{\pi} = 0,$$
$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx\,dx = \frac{1}{\pi}\left[\frac{1}{k}x\sin kx\right]_{-\pi}^{\pi} - \frac{1}{k}\int_{-\pi}^{\pi}\sin kx\,dx$$
$$= \frac{1}{\pi}\left[\frac{1}{k}x\sin kx + \frac{1}{k^2}\cos kx\right]_{-\pi}^{\pi} = \frac{2}{\pi k^2},$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx\,dx = \frac{1}{\pi}\left[-\frac{1}{k}x\cos kx\right]_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi}\cos kx\,dx$$
$$= \frac{1}{\pi}\left[\frac{1}{k}x\sin kx + \frac{1}{k^2}\sin kx\right]_{-\pi}^{\pi} = 0.$$

以上より，

$$a_0 = 0, \quad a_k = \frac{2}{\pi k^2}, \quad b_k = 0. \qquad \cdots\cdots(\text{答})$$

b)
$$f_2(x) = \begin{cases} -1, & -\pi \le x < -\frac{\pi}{2} \\ 1, & -\frac{\pi}{2} \le x < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} \le x < \pi \end{cases}$$
$$f_2(x+2\pi) = f_2(x).$$

この $f_2(x)$ に対して，

$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2}(-1)\,dx + \int_{-\pi/2}^{\pi/2}1\,dx + \int_{\pi/2}^{\pi}(-1)\,dx\right) = \frac{1}{\pi}\left(-\frac{\pi}{2} + \pi - \frac{\pi}{2}\right) = 0,$$
$$a_k = \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2}(-\cos kx)\,dx + \int_{-\pi/2}^{\pi/2}\cos kx\,dx + \int_{\pi/2}^{\pi}(-\cos kx)\,dx\right)$$
$$= \frac{1}{\pi}\left[-\frac{1}{k}\sin kx\right]_{-\pi}^{-\pi/2} + \left[\frac{1}{k}\sin kx\right]_{-\pi/2}^{\pi/2} + \left[-\frac{1}{k}\sin kx\right]_{\pi}^{\pi/2} = \frac{2}{\pi k}\sin\frac{\pi k}{2},$$
$$b_k = \frac{1}{\pi}\left(\int_{-\pi}^{-\pi/2}(-\sin kx)\,dx + \int_{-\pi/2}^{\pi/2}\sin kx\,dx + \int_{\pi/2}^{\pi}(-\sin kx)\,dx\right)$$
$$= \frac{1}{\pi}\left[\frac{1}{k}\cos kx\right]_{-\pi}^{-\pi/2} + \left[-\frac{1}{k}\cos kx\right]_{-\pi/2}^{\pi/2} + \left[\frac{1}{k}\cos kx\right]_{\pi}^{\pi/2} = -\frac{2}{\pi k}\cos k\pi.$$

以上より，

$$a_0 = 0, \quad a_k = \frac{2}{\pi k}\sin\frac{\pi k}{2}, \quad b_k = -\frac{2}{\pi k}\cos\pi k. \qquad \cdots\cdots(\text{答})$$

c) $f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$ より,

$$f(x)^2 = \frac{{a_0}^2}{4} + \sum_{k=0}^{\infty}({a_k}^2 \cos^2 kx + {b_k}^2 \sin^2 kx) + \sum_{k \neq l} a_k b_l \cos kx \sin kx$$

であり,

$$\int_{-\pi}^{\pi} \frac{{a_0}^2}{4}\, dx = \frac{\pi}{2}{a_0}^2,$$

$$\int_{-\pi}^{\pi} \cos^2 kx\, dx = \int_{-\pi}^{\pi} \frac{1+\cos 2kx}{2}\, dx = \left[\frac{x + \frac{1}{k}\sin 2kx}{2}\right]_{-\pi}^{\pi} = \pi$$

$$\int_{-\pi}^{\pi} \sin^2 kx\, dx = \int_{-\pi}^{\pi} \frac{1-\cos 2kx}{2}\, dx = \left[\frac{x - \frac{1}{k}\sin 2kx}{2}\right]_{-\pi}^{\pi} = \pi$$

$$\int_{-\pi}^{\pi} \cos kx \sin lx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}(\sin(k+l)x - \sin(k-l)x)\, dx$$

$$= \frac{1}{2}\left[-\frac{1}{k+l}\cos(k+l)x - \frac{1}{k-l}\cos(k-l)x\right]_{-\pi}^{\pi} = 0$$

であることに注意すれば,

$$\int_{-\pi}^{\pi} f(x)^2\, dx$$

$$= \int_{-\pi}^{\pi} \frac{{a_0}^2}{4}\, dx + \sum_{k=0}^{\infty}\left({a_k}^2 \int_{-\pi}^{\pi} \cos^2 kx\, dx + {b_k}^2 \int_{-\pi}^{\pi} \sin^2 kx\, dx\right) + \sum_{k \neq l} a_k b_l \int_{-\pi}^{\pi} \cos kx \sin lx\, dx$$

$$= \frac{{a_0}^2}{2}\pi + \sum_{k=0}^{\infty}({a_k}^2 \cdot \pi + {b_k}^2 \cdot \pi) + 0.$$

よって,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\, dx = \frac{{a_0}^2}{2} + \sum_{k=0}^{\infty}({a_k}^2 + {b_k}^2)$$

を得る. (おわり)

2) a)

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad G(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\Omega)$$

$g(t)$ をフーリエ変換すると

$$\begin{aligned}
\int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,dt &= \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\delta(t-kT)e^{-i\omega t}\,dt \\
&= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(t-kT)e^{-i\omega t}\,dt \\
&= \sum_{k=-\infty}^{\infty} e^{-i\omega kT} = \sum_{k=-\infty}^{\infty} e^{-ik(\omega T)} \\
&= 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega T - 2\pi k) \\
&= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\Big(\omega - \frac{2\pi}{T}k\Big).
\end{aligned}$$

よって

$$\Omega = \frac{2\pi}{T}. \qquad \cdots\cdots(答)$$

b)

$$\begin{aligned}
X_s(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega-\omega')G(\omega')\,d\omega' \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega-\omega')\cdot\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\Big(\omega' - \frac{2\pi}{T}k\Big)\,d\omega' \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X(\omega-\omega')\delta\Big(\omega' - \frac{2\pi}{T}k\Big)\,d\omega' \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\Big(\omega - \frac{2\pi}{T}k\Big). \qquad \cdots\cdots(答)
\end{aligned}$$

c) サンプリング定理の仮定を満たしていないことから, エイリアシングが起こり, 元の波形を再現できない.